

# Worst Case Number of Terms In Symmetric Multiple-Valued Functions\*

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## Abstract

*A symmetric multiple-valued function can be realized as the disjunction of fundamental symmetric functions. A simpler disjunction can be formed when the latter functions combine in the same way that minterms combine to form simpler product terms for sum-of-products expressions. We solve a problem posed by Muzio [7], who seeks the worst case symmetric function in the sense that the maximum number of fundamental symmetric functions is needed. The worst case is presented for 3- and 4-valued systems in [7], but the case for general radix is left open. We solve this problem for general radix, and show that the ratio of the maximum size of the disjunction to the total number of fundamental symmetric functions approaches one-half as the number of variables increases.*

## 1. Introduction

Since the 1940's, considerable study has been done on the problem of logic design. The classic problem has been to build a given function from component functions in some minimal fashion. That is, the components represent "off-the-shelf" functions available in sufficient numbers. The idea is to use as few of these components as possible. Often restrictions apply. For example, the realization may be in the form of a "sum-of-products" expression. In this case, the goal is to minimize the number of product terms.

Because they occur so often in design problems, symmetric functions have received considerable scrutiny. The long history of study of symmetric functions in binary systems, has, to a lesser extent, occurred in

multiple-valued systems [2-4, 6-8]. Recently, Muzio [7] has studied the realization of general multiple-valued symmetric functions from fundamental symmetric functions. Specifically, any arbitrary symmetric function can be realized as the disjunction of fundamental symmetric functions. Some of these functions may combine, realizing a savings. The object of the design is to realize the given function with as few combinations as possible. The problem considered in [7] is the worst case. That is, certain functions will require the maximum number of combinations, and it is desired to find this maximum number. Muzio [7] computes the maximum for 3- and 4-valued systems and conjectures the values for general  $m$ -valued systems. We solve the general problem and show that, as in the 3- and 4-valued case, the ratio of the maximum number of components to the total number of fundamental symmetric functions approaches one-half as the number of variables  $n$  increases.

We consider  $m$ -valued functions of  $n$   $m$ -valued variables. Let  $R = \{0, 1, \dots, m-1\}$ , where  $m \geq 2$ . Let  $X = \{x_1, x_2, \dots, x_n\}$  be a set of  $n$  variables, where  $x_i$  takes on values from  $R$ . A function  $f$  is a mapping  $f : R^n \rightarrow R$ .

$f$  is symmetric in  $x_i$  and  $x_j$  iff  $f(x_1, \dots, x_i, \dots, x_j, \dots, x_n) = f(x_1, \dots, x_j, \dots, x_i, \dots, x_n)$ .  $f$  is (totally) symmetric iff it is symmetric in all pairs of variables. For example, the function  $f(x_1, x_2, x_3) = \max\{x_1, x_2, x_3\}$  (i.e.  $\max\{x_1, x_2, x_3\}$  takes on the maximum of the values assigned to  $x_1, x_2$ , and  $x_3$ ) is a symmetric function. Let  $\alpha = \{\alpha_0, \alpha_1, \dots, \alpha_{m-1}\}$ , where  $0 \leq \alpha_i \leq n$  and  $\alpha_0 + \alpha_1 + \dots + \alpha_{m-1} = n$ . A fundamental symmetric function (FSF) is a symmetric function that has the

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value  $m-1$  if, for each  $i$ ,  $0 \leq i \leq m-1$ ,  $\alpha_i$  of the variables are  $i$  and has the value 0 otherwise. It is convenient to represent an FSF as  $f_{\alpha_0 \alpha_1 \dots \alpha_{m-1}}(X)$ . For example,  $f_{1022}(X)$  is a 4-valued 5-variable FSF that is 3 if one of the variables is 0, zero are 1, two are 2, and two are 3 and is 0 otherwise.

Let  $+$  denote the *max* operation. Specifically, for  $a, b \in R$ ,  $a + b = \max\{a, b\}$ .  $+$  is associative and extends to more than two operands in a natural way. For example,  $a + b + c = \max\{a, b, c\}$ . Let concatenation denote the *min* operation. Specifically, for  $a, b \in R$ ,  $a \cdot b = \min\{a, b\}$ .

FSF's can be used to form arbitrary symmetric functions as follows. Given a symmetric function,  $f(X)$ , it can always be expressed as  $f(X) = 1g_1(X) + 2g_2(X) + \dots + (m-1)g_{m-1}(X)$ , where  $g_i(X)$  is the sum (max) of FSF's, for  $1 \leq i \leq m-1$ . The following from Muzio [7] is a description of the problem.

*Given this potential simplification, our interest is in the worst case - how large can the sum become without being amenable to any simplification. ... we consider how many FSF's could be included in a disjunction, where no two of them can be combined into a simpler form.*

## 2. The Number of Fundamental Symmetric Functions

We repeat the calculation of the number of fundamental symmetric functions given in [7], except that we use generating functions. This introduces our approach in solving the general problem. Let  $P(m, n)$  be the number of  $m$ -valued FSF's on  $n$  variables, and let  $P[w, x]$  be the generating function for  $P(m, n)$ . That is,

$$\begin{aligned} P[w, x] = & P(0,0) + \left[ P(1,0) + P(1,1)x + P(1,2)x^2 + \right. \\ & \dots \left. \right] w + \left[ P(2,0) + P(2,1)x + P(2,2)x^2 + \dots \right] w^2 + \\ & \dots + P(m, n)x^m w^m + \dots \end{aligned}$$

Here,  $w$  tracks  $m$ , while  $x$  tracks  $n$ . We seek a closed-form expression for  $P[w, x]$ .

Since each FSF,  $f(X)$ , is uniquely specified by choosing values for  $\alpha_0, \alpha_1, \dots$ , and  $\alpha_{m-1}$  in  $f(X) =$

$f_{\alpha_0 \alpha_1 \dots \alpha_{m-1}}$ , we count the number of FSF's by counting the number of ways to assign these values. The ways to assign a value to one  $\alpha_i$  are enumerated in the generating function

$$h[x] = 1 + x + x^2 + \dots + x^j + \dots = \frac{1}{1-x}.$$

Each  $x^j$  term occurs with a coefficient of 1 because there is only one way to choose  $\alpha_i$  to have the value  $j$ . Variable  $x$  is used because a choice of  $j$  for a single  $\alpha_i$  contributes  $j$  to  $n$ , which is tracked by  $x$ . Further, a choice of a value for a single  $\alpha_i$  contributes 1 to  $m$ , and, so we represent the complete contribution of a choice of a single  $\alpha_i$  as  $h[x]w$ . Similarly, a choice for  $m$   $\alpha$ 's is expressed by  $h^m[x]w^m$ . Therefore, the generating function for the number of FSF's,  $P(m, n)$  is

$$\begin{aligned} P[w, x] = & 1 + h[x]w + h^2[x]w^2 + \\ & \dots + h^m[x]w^m + \dots \end{aligned} \quad (1)$$

Here, the  $h^m[x]w^m$  term enumerates all  $m$ -valued FSF's. Further,  $h^m[x]$  enumerates the ways to choose  $n$  objects from  $m$  objects with unlimited repetition (e.g. Liu [5, pp. 31-32]), which is  $\binom{m+n-1}{n}$ . This proves

**Theorem 1:** [7] The number of  $m$ -valued fundamental symmetric functions on  $n$  variables is

$$P(m, n) = \binom{m+n-1}{n}. \quad (2)$$

From (1), we can write a closed-form expression for  $P[w, x]$  as

$$P[w, x] = \frac{1}{1 - h[x]w}.$$

Substituting for  $h[x]$  and rearranging yields,

$$P[w, x] = \frac{1-x}{1-x-w}. \quad (3)$$

We prefer this form over (2) because of the ease with which values of  $P(m, n)$  can be obtained by a symbolic mathematics package. For example, using MACSYMA to generate a Taylor series expansion of (3), we obtain, in a single command line of 37 keystrokes, the expression in Fig. 1. These values agree, as they should, with the values in Table 1 of Muzio [7], which shows the number of FSF's.

$$\begin{aligned}
P[w, x] = & 1 + (1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + \dots)w \\
& + (1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + 8x^7 + 9x^8 + 10x^9 + 11x^{10} + \dots)w^2 \\
& + (1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5 + 28x^6 + 36x^7 + 45x^8 + 55x^9 + 66x^{10} + \dots)w^3 \\
& + (1 + 4x + 10x^2 + 20x^3 + 35x^4 + 56x^5 + 84x^6 + 120x^7 + 165x^8 + 220x^9 + 286x^{10} + \dots)w^4 + \dots
\end{aligned}$$

Figure 1. Generating function for the number of fundamental symmetric functions.

The generating function of (1) allows a direct computation of a closed-form approximation to  $P(m, n)$  as  $n \rightarrow \infty$ . That is, (1) gives the generating function for  $P(m, n)$  for fixed  $m$ . We use this later to solve for the fraction of FSF's that are in a maximal form as  $n \rightarrow \infty$ . Let  $f(n) \sim g(n)$  mean  $\lim_{n \rightarrow \infty} f(n)/g(n) = 1$ . Applying Theorem 4 of Bender [1] to the generating function,  $w^m/(1-x)^m$ , where  $m$  is a fixed integer, yields,

$$P(m, n) \sim \frac{n^{m-1}}{(m-1)!}. \quad (4)$$

It is important to note that this approximation is based on the assumption that  $m$  is a fixed, while  $n$  is an arbitrarily large. (4) can also be obtained by expressing (2) in factorials and using Stirling's approximation.

### 3. Worst Case Disjunctions of Fundamental Symmetric Functions

Two FSF's,  $f_{\alpha_0 \alpha_1 \dots \alpha_{m-1}}$  and  $f_{\beta_0 \beta_1 \dots \beta_{m-1}}$  are *adjacent* iff there exists an  $i$ , where  $0 \leq i \leq m-2$ , such that either 1)  $\alpha_i = \beta_i + 1$  and  $\alpha_{i+1} = \beta_{i+1} - 1$  or 2)  $\alpha_i = \beta_i - 1$  and  $\alpha_{i+1} = \beta_{i+1} + 1$ , while  $\alpha_j = \beta_j$  for  $j \notin \{i, i+1\}$ . For example,  $f_{211}$  is adjacent to  $f_{202}$ .

Adjacency can be represented by an *adjacency graph*, in which nodes correspond to FSF's and edges correspond to adjacency. For example, Fig. 2a represents all of the adjacencies between  $m = 3$ -valued FSF's with  $n = 4$  variables. In writing a node label, we discard the  $f$ , retaining only the subscript. For example, there is an edge between nodes 211 and 202 because  $f_{211}$  is adjacent to  $f_{202}$ . In the case of the adjacency graph for 3-valued FSF's on 4 variables, there are, in all,  $\binom{3+4-1}{4} = 15$  nodes and 20 adjacencies. Fig. 2b shows the adjacency graph for all 4-valued FSF's on 3 variables. In this case, there are  $\binom{4+3-1}{3} = 20$  nodes and 30 adjacencies. Let  $G(m, n)$  denote the adjacency graph for  $m$ -valued FSF's on  $n$ -variables.

The problem posed at the end of the Introduction can be stated as follows.

*What is the maximum number of ways  $M(m, n)$  to select  $m$ -valued FSF's on  $n$  variables such that no two are adjacent?*

This problem has an analog in graph theory. That is, the concept of adjacency, as defined here, is identical to the graph theory concept of adjacency; two nodes are *adjacent* iff they are joined by an edge. The problem is therefore identical to the problem of selecting the largest number of nodes in an adjacency graph such that no two are adjacent. For example, in the adjacency graph for 3-valued FSF's on 4 variables, this number is  $M(3, 4) = 9$ . The large dots in Fig. 2a represent the (unique) largest set of nonadjacent nodes. Similarly, in the adjacency graph for 4-valued FSF's on 3 variables of Fig. 2b, the largest number of nonadjacent nodes is  $M(4, 3) = 10$ . In this case, there are two largest sets of nonadjacent nodes, the one shown by large dots and another represented by smaller dots.

A graph in  $G$  is *bipartite* iff the nodes of  $G$  can be partitioned into two nonempty subsets  $G_1$  and  $G_2$ , such that no edge in  $G$  is incident to nodes exclusively in  $G_1$  or exclusively in  $G_2$ . That is, in a bipartite graph, all edges are incident to one node in  $G_1$  and one node in  $G_2$ . For example, the two graphs in Fig. 2 are bipartite.

We show that *any* adjacency graph  $G(m, n)$  is bipartite, as follows. Given  $\alpha = \{\alpha_0, \alpha_1, \dots, \alpha_{m-1}\}$ , divide  $\alpha$  into two subsets,  $\alpha_0 = \{\alpha_0, \alpha_2, \alpha_4, \dots\}$  and  $\alpha_1 = \{\alpha_1, \alpha_3, \alpha_5, \dots\}$ . Let  $n_0(\alpha) = \alpha_0 + \alpha_2 + \alpha_4 + \dots$  and  $n_1(\alpha) = \alpha_1 + \alpha_3 + \alpha_5 + \dots$ . It follows that  $n_0(\alpha) + n_1(\alpha) = n$ . Note that if FSF's  $f_{\alpha_0 \alpha_1 \dots \alpha_{m-1}}$  and  $f_{\beta_0 \beta_1 \dots \beta_{m-1}}$  are adjacent, then either 1)  $n_0(\alpha), n_1(\alpha) = n_0(\beta) + 1, n_1(\beta) - 1$  or 2)  $n_0(\alpha), n_1(\alpha) = n_0(\beta) - 1, n_1(\beta) + 1$ . For example, with FSF's  $f_{211}$  ( $f_\alpha$ ) and  $f_{202}$  ( $f_\beta$ ),  $n_0(\alpha), n_1(\alpha) = 3, 1$  and  $n_0(\beta), n_1(\beta) = 4, 0$ . Here,  $n_0(\alpha), n_1(\alpha) = n_0(\beta) - 1, n_1(\beta) + 1$ .

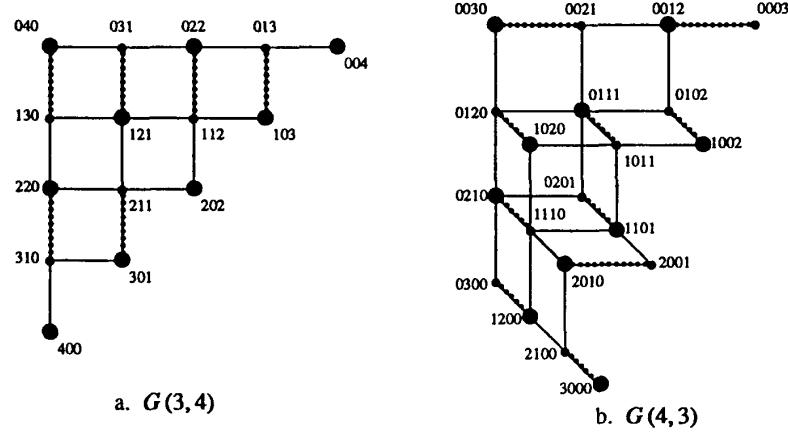


Figure 2. Adjacency graphs for 3-valued 4-variable and 4-valued 3-variable FSF's.

Given FSF  $f_\alpha$ , define its *parity* according to the parity of  $n_0(\alpha)$  and  $n_1(\alpha)$ . That is, when  $n$  is even, either 1)  $n_0(\alpha)$  is odd and  $n_1(\alpha)$  is odd or 2)  $n_0(\alpha)$  is even and  $n_1(\alpha)$  is even. In this case, we say that the parity of  $f_\alpha$  as OO (for odd/odd) or EE (for even/even), respectively. For example,  $f_{211}$  has parity OO, while  $f_{202}$  has parity EE. When  $n$  is odd, then either 1)  $n_0(\alpha)$  is odd and  $n_1(\alpha)$  is even or 2)  $n_0(\alpha)$  is even and  $n_1(\alpha)$  is odd. In this case, we say that the parity of  $f_\alpha$  is OE or EO, respectively.

For a fixed  $m$  and  $n$ , we say that two FSF's have *opposite parity* if, for even  $n$ , one has parity OO and the other EE, or, for odd  $n$ , one has parity OE and the other has parity EO. For example, adjacent FSF's  $f_{211}$  and  $f_{202}$  have opposite parity. The important observation of this discussion is that when two FSF's are adjacent, they have opposite parity. That is, if two FSF's are adjacent, their  $\alpha$  values are the same except for the transfer of a 1 from one  $\alpha_i$  to a neighbor. This transfer changes the parity. Thus, given an adjacency graph in which nodes are partitioned according to the two parities, edges can only occur between parts. This proves

**Lemma 1:** The adjacency graph  $G(m,n)$  for  $m$ -valued FSF's on  $n$  variables is bipartite.

It is interesting to note also that the partition of nodes in an adjacency graph is unique to within an interchange of the two parts. This follows because an adjacency graph is *connected*; there is a path from any node to any other node. Therefore, as we progress from one node to any other, we prescribe the side on which a node resides by whether it is an even distance or an odd

distance from the start node.

It is tempting to believe that, because the graph is bipartite, the maximum number of nonadjacent nodes is the number of nodes in the larger part. However, this is not always true; there are bipartite graphs in which the largest number of nonadjacent nodes exceeds the number of nodes in the larger part.

Consider a bipartite graph  $G$ , with parts  $G_1$  and  $G_2$  which are the subsets of nodes within which no two nodes are connected by an edge. A *matching* in this graph is a subset of edges such that no two edges are incident to the same node. That is, in a matching, the edges are disjoint. For example, in Fig. 2, the edges shown as rippled lines define a matching. A *complete matching* of  $G_1$  into  $G_2$  is a matching such that there is an edge incident to every node in  $G_1$ . For example, the matching shown in Fig. 2 is complete. We will prove

**Lemma 2:** The adjacency graph  $G(m,n)$  of  $m$ -valued FSF's on  $n$  variables contains a complete matching.

The value of Lemma 2 is that it allows us to conclude that the maximum number of nonadjacent nodes in a bipartite adjacency graph is the number of nodes in the larger part (in a bipartite adjacency graph, if both parts are of equal size, either is the "larger" part). Consider a largest set  $S = \{\alpha_1, \alpha_2, \dots, \alpha_k\}$  of nonadjacent nodes in which a nonempty subset  $S'$  of  $S$  is in the smaller part. Because there exists a complete matching, we can achieve another largest set of nonadjacent nodes by moving each  $\alpha_i$  to the larger side using the complete

matching. That is, the complete matching establishes a one-to-one mapping of nodes in the smaller subgraph to nodes in the larger subgraph. In replacing each node in  $S'$  with its counterpart, as specified by the complete matching, we achieve another largest set of nonadjacent nodes in which all nodes are in the larger part. It follows that the maximum number of nonadjacent  $m$ -valued FSF's on  $n$  variables is the number of nodes in the larger part of the bipartite adjacency graph  $G(m, n)$ . The main result of this paper is achieved by counting the latter. In the process of doing this, we prove Lemma 2.

Let  $\alpha = \{\alpha_0, \alpha_1, \dots, \alpha_{m-1}\}$ , and let  $\beta = \{\beta_0, \beta_1, \dots, \beta_{m-1}\}$ . Consider a set  $\Gamma_{m,n}$  of edges in adjacency graph  $G(m, n)$  defined as follows. Edge  $(\alpha, \beta) \in \Gamma_{m,n}$  iff the following conditions hold. For  $\alpha$ ,

- 1) for some even  $i < m-1$ ,  $\alpha_i$  is odd,
- 2) for all even  $j < i$ ,  $\alpha_j$  is even, and
- 3) for all odd  $j < i$ ,  $\alpha_j$  is 0,

and for  $\beta$ ,

- 1)  $\beta_i = \alpha_i - 1$ ,
- 2)  $\beta_{i+1} = \alpha_{i+1} + 1$ , and
- 3)  $\beta_j = \alpha_j$  for  $j \notin \{i, i+1\}$ .

For example, for 3-valued FSF's on 4 variables,  $\Gamma_{3,4}$  consists of  $\{(103, 013), (112, 022), (121, 031), (130, 040), (301, 211), (310, 220)\}$ , and for 4-valued FSF's on 3 variables,  $\Gamma_{4,3}$  consists of  $\{(1002, 0102), (1011, 0111), (1020, 0120), (1101, 0201), (1110, 0210), (1200, 0300), (3000, 1200), (0012, 0003), (0030, 0021), (2010, 2001)\}$ .  $\Gamma_{m,n}$  corresponds to the set of edges in Fig. 2 with rippled lines.

By construction, no edge in  $G(m, n)$  shares a node with any other edge in  $G(m, n)$ . That is, all nodes,  $\alpha$  and  $\beta$ , are distinct. Therefore,  $\Gamma_{m,n}$  is a matching. We show that  $\Gamma_{m,n}$  is a complete matching by showing that all nodes in one part of the bipartite graph are incident to an edge in  $G(m, n)$ . Specifically, we show that nodes that correspond to FSF's with parity  $\bar{P}O$  are incident to some edge in  $G(m, n)$ , where  $P$  is the parity of  $n$  ( $P = O$  if  $n$  is odd and  $P = E$  if  $n$  is even) and  $\bar{P}$  is the complement of  $P$ .

Consider a node  $\gamma$  with parity  $\bar{P}O$ . Because of the  $O$  in  $\bar{P}O$ , there is a smallest odd  $i$  such that  $\gamma_i$  is nonzero. Let  $j$  be the smallest even index where  $\gamma_j$  is odd, if indeed such a  $j$  exists. If it does not exist, let  $j = m+1$ . If  $j < i$ , then  $\gamma$  is incident to an edge in  $\Gamma_{m,n}$ ; it satisfies the definition for  $\alpha$  in  $(\alpha, \beta)$ . If  $j > i$ , then  $\gamma$  is also incident to an edge in  $\Gamma_{m,n}$ ; it satisfies

the definition for  $\beta$  in  $(\alpha, \beta)$ , where  $\beta = \gamma$  and  $\alpha_k = \gamma_k$  except that  $\alpha_{i-1} = \gamma_{i-1} + 1$  and  $\alpha_i = \gamma_i - 1$ .

Since nodes of the form  $\bar{P}O$  are incident to an edge in  $\Gamma_{m,n}$ , and such nodes represent *all* nodes on one side of the adjacency graph, then the matching  $\Gamma_{m,n}$  is complete. This proves Lemma 2. It follows that there are at least as many nodes in the other (larger) part. This proves

**Theorem 2:** The maximum number of nonadjacent  $m$ -valued FSF's on  $n$  variables is the number of FSF's with parity  $PE$ , where  $P$  is the parity of  $n$  ( $E$  if  $n$  is even and  $O$  if  $n$  is odd).

Now consider the calculation of the maximum number of nonadjacent  $m$ -valued FSF's on  $n$  variables. Our approach is to calculate the number of nodes in the smaller part of this bipartite graph and to subtract this from the total number of nodes (total number of FSF's). We note that the number of nodes in the smaller part is equal to the number of edges in  $G(m, n)$ , which is the number of ways to choose  $\alpha$  in the definition of  $\Gamma_{m,n}$ . That is,  $\alpha$  has the form  $\alpha_0 0 \alpha_2 0 \dots \alpha_i \alpha_{i+1} \dots \alpha_m$ , where  $i$  is the smallest even  $i$  such that  $\alpha_i$  is odd, and where  $\alpha_0, \alpha_2, \dots, \alpha_{i-2}$  are even. Again, we use generating functions. The generating function for 00, 20, 40, ... is

$$h_E[x] = 1 + x^2 + x^4 + \dots = \frac{1}{1 - x^2}.$$

Here,  $x$  is used because the choices 00, 20, 40, ... contribute 0, 2, 4, ..., respectively, to  $n$ . If we track  $m$  using  $w$ , then we have

$$h_E[x]w^2.$$

Prior to the first odd  $i$  such that  $\alpha_i$  is odd, there can be no, one, two, three, etc. pairs of the form 00, 20, 40, ..., and these choices correspond to

$$1 + h_E[x]w^2 + h_E^2[x]w^4 + \dots + h_E^i[x]w^{2i} + \dots = \frac{1}{1 - h_E[x]w^2}.$$

With a choice for  $\alpha_i$  of an odd value, the generating function for  $\alpha_i$  is

$$h_O[x]w = (x + x^3 + x^5 + \dots)w = \frac{xw}{1 - x^2}.$$

Since  $i < m-1$ , at the least,  $\alpha_{i+1}$  exists. However, there could be others. The generating function for the

elements beyond  $\alpha_i$  is

$$h[x]w + h^2[x]w^2 + h^3[x]w^3 + \dots = \frac{h[x]w}{1-h[x]w}.$$

Therefore, the generating function for the number of nodes in the smaller part of the bipartite graph for  $m$ -valued FSF's on  $n$  variables is

$$\frac{h[x]h_O[x]w^2}{(1-h[x]w)(1-h_E[x]w^2)}.$$

As mentioned previously, the maximum number of nonadjacent FSF's is the total number of FSF's less the number on the smaller side of the bipartite graph. Thus, the generating function,  $M[w, x]$ , for  $M(m, n)$ , the maximum number of ways to select  $m$ -valued FSF's on  $n$  variables is

$$M[w, x] = \frac{1}{1-h[x]w} - \frac{h[x]h_O[x]w^2}{(1-h[x]w)(1-h_E[x]w^2)}.$$

Substituting for  $h[x]$ ,  $h_E[x]$ , and  $h_O[x]$  and rearranging yields

$$M[w, x] = \left[ \frac{1}{(1-x-w)} \right] \left[ 1-x - \frac{w^2 x}{1-x^2-w^2} \right]. \quad (5)$$

Using MACSYMA to generate a Taylor series expansion of (5), we obtain, in a single command line of 54 keystrokes, the equation shown in Fig. 3.

Table I on the next page shows the values of  $M(m, n)$  and  $P(m, n)$  for  $2 \leq m \leq 10$  and  $2 \leq n \leq 8$ . Each entry has the form  $M(m, n)/P(m, n)$ . The values of  $M(m, n)$  in this table agree with the conjectured values in Table 4 of Muzio [7]. An examination of this table shows that the values of  $M(m, n)$  are

$$\begin{aligned} M[w, x] = & 1 + (1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + \dots)w \\ & + (1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + 4x^6 + 4x^7 + 5x^8 + 5x^9 + 6x^{10} + \dots)w^2 \\ & + (1 + 2x + 4x^2 + 6x^3 + 9x^4 + 12x^5 + 16x^6 + 20x^7 + 25x^8 + 30x^9 + 36x^{10} + \dots)w^3 \\ & + (1 + 2x + 6x^2 + 10x^3 + 19x^4 + 28x^5 + 44x^6 + 60x^7 + 85x^8 + 110x^9 + 146x^{10} + \dots)w^4 \\ & + (1 + 3x + 9x^2 + 19x^3 + 38x^4 + 66x^5 + 110x^6 + 170x^7 + 255x^8 + 365x^9 + 511x^{10} + \dots)w^5 + \dots. \end{aligned}$$

Figure 3. Generating function for the number of noncombinable FSF's in a symmetric function.

approximately one-half of the values of  $P(m, n)$ . Further, as  $n$  becomes larger, the approximation becomes better. Indeed, Muzio [7] conjectures that  $M(m, n)/P(m, n) \rightarrow 1/2$  as  $n \rightarrow \infty$ . We now prove this.

**Theorem 3:** The ratio of the maximum number of nonadjacent  $m$ -valued FSF's on  $n$  variables to the total number approaches  $1/2$  as  $n$  becomes arbitrarily large.

Consider the complete matching,  $\Gamma_{m,n}$ . Our approach is to count the number of nodes  $Q(m, n)$  not incident to an edge in this matching, and to show that  $Q(m, n)$  is a vanishingly small fraction of the total number of nodes  $P(m, n)$  as  $n \rightarrow \infty$  for fixed  $m$ . This shows that, for large  $n$ , almost all nodes in the larger part are matched to nodes in the smaller part, and the maximum number of nonadjacent nodes is close to one-half of the total number of FSF's. Nodes that are *not* incident to any node have the form

$$\begin{aligned} \alpha_0 0 \alpha_2 0 \alpha_4 0 \dots \alpha_{m-2} 0 & \quad \text{for even } m \text{ or,} \\ \alpha_0 0 \alpha_2 0 \alpha_4 0 \dots \alpha_{m-3} 0 \alpha_{m-1} & \quad \text{for odd } m, \end{aligned}$$

where  $\alpha_i$  is even for even  $i$ , except for  $\alpha_{m-1}$ , which is unrestricted. The generating function  $Q_{m_E}[x]$  for  $Q(m, n)$  when  $m$  is even is

$$Q_{m_E}[x] = \frac{1}{(1-x^2)^{m/2}}, \quad (6)$$

while the generating function  $Q_{m_O}[x]$  for  $Q(m, n)$  when  $m$  is odd is

$$Q_{m_O}[x] = \frac{1}{(1-x^2)^{(m-1)/2}(1-x)} = \frac{1+x}{(1-x^2)^{(m+1)/2}}. \quad (7)$$

For example, in the middle expression of (7), the factor  $1/(1-x^2)$  represents the ways to choose a *single* even

$m = n$	2	3	4	5	6	7	8	9	10
2	2/3	4/6	6/10	9/15	12/21	16/28	20/36	25/45	30/55
3	2/4	6/10	10/20	19/35	28/56	44/84	60/120	85/165	110/220
4	3/5	9/15	19/35	38/70	66/126	110/210	170/330	255/495	365/715
5	3/6	12/21	28/56	66/126	126/252	236/462	396/792	651/1287	1001/2002
6	4/7	16/28	44/84	110/210	236/462	472/924	868/1716	1519/3003	2520/5005
7	4/8	20/36	60/120	170/330	396/792	868/1716	1716/3432	3235/6435	5720/11440
8	5/9	25/45	85/165	255/495	651/1287	1519/3003	3235/6435	6470/12870	12190/24310

Table I.  $M(m, n)/P(m, n)$  = maximum number of nonadjacent FSF's to the total number of FSF's.

value for  $\alpha_i$ , while  $1/(1-x^2)^{(m-1)/2}$  represents the ways to choose all terms,  $\alpha_0, \alpha_2, \dots$ , and  $\alpha_{m-3}$  each with an even value. The factor  $1/(1-x)$  represents the ways to choose  $\alpha_{m-1}$ .

We show that  $Q(m, n)$  represents a vanishingly small fraction of  $P(m, n)$ , the number of FSF's, as  $n \rightarrow \infty$ , for even  $m$ . For odd  $m$ , the proof is similar. The generating function for even  $m$ , (6), has only a single  $x^2$  term. Thus, the power series expansion of this shows the coefficient of odd powers of  $x$  are 0. Thus,  $Q(m, n) = 0$ , for odd  $m$ . For even  $m$ , we can derive the coefficient as

$$Q(m, n) = \binom{m/2 + n/2 - 1}{n/2} = \frac{(m/2 + n/2 - 1)!}{(m/2 - 1)! (n/2)!},$$

from which we can write the asymptotic approximation

$$Q(m, n) \sim \frac{n^{(m/2 - 1)}}{(m/2 - 1)! 2^{(m/2 - 1)}}. \quad (8)$$

From this and (4), the asymptotic approximation for  $P(m, n)$ , the total number of FSF's, we can write

$$\frac{Q(m, n)}{P(m, n)} \sim \frac{(m - 1)! n^{-m/2}}{(m/2 - 1)! 2^{(m/2 - 1)}}.$$

From this expression, we have

$$\lim_{n \rightarrow \infty} \frac{Q(m, n)}{P(m, n)} = 0.$$

Therefore, for large  $n$  (and even  $m$ ), the fraction of

nodes *not* involved in a matching is close to 0. A similar derivation exists for odd  $m$ , proving Theorem 2.

#### 4. Concluding Remarks

We have solved a problem posed by Muzio [7] on the number of nonadjacent fundamental symmetric functions. This represents the worst case symmetric function, the one with the largest number of uncombinable fundamental symmetric functions. This was a two step process. The first step was to prove that this number is equal to the number of nodes in the larger part of a bipartite adjacency graph. The second step was to apply generating functions to accomplish the enumeration. It is shown that the worst case requires approximately one half the number of fundamental symmetric functions. This has an analog in binary sum-of-products expressions. That is, the binary function with the most number of product terms (the exclusive OR function) requires exactly one half the total number of minterms.

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